AN INJECTIVITY RADIUS ESTIMATE IN TERMS OF METRIC SPHERE

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Abstract. In this paper we prove that if a point p in a complete Riemannian manifold is not a cut point of any point whose distance to p is r, then the injectivity radius of p is strictly large than r. As a corollary we give a positive answer to a problem raised by Z. Sun and J. Wan.

This paper is to answer a question asked by Z. Sun and J. Wan in [2]. Let M be a complete noncompact Riemannian manifold, and let i_p denote the injectivity radius at p of M. Let

$$i(p,r)=\min\{i_x : \forall\, x\in M \text{ s.t. } \mathrm{d}(x,p)=r\},$$

where $\mathrm{d}(x,p)$ is the distance between two points x and p. According to [2], they defined a number $\alpha(M)$ to be

$$\alpha(M)=\liminf_{r\to\infty}\frac{i(p,r)}{r},$$

which is called the *injectivity radius growth* of M. Because in the definition of $\alpha(M)$ r goes to infinity and the distance from p to any other fixed point is a definite finite number, it can be seen directly (see also a proof in [2]) that $\alpha(M)$ is not depending on p. One of their questions in [2] is the following

Question 1 ([2]). *For a complete noncompact manifold M, can one prove that every geodesic $\gamma:(-\infty,+\infty)\to M$ is a line as long as $\alpha(M)>1$?*

In other words, they asked that whether the injectivity radius of every point in M is infinity when $\alpha(M)>1$? A positive answer of Question 1 directly follows from Proposition 2 below.

Proposition 2. *Let M be a complete Riemannian manifold and $p\in M$. If for some $r>0$, p is not a cut point of any point x such that $\mathrm{d}(x,p)=r$, then the injectivity radius i_p at $p>r$.*

Remark 3. The point in proving Proposition 2 is to show that the minimal geodesics for p to points in the metric sphere $S_r(p)=\{x\in M:\mathrm{d}(p,x)=r\}$ covers the whole ball $B_r(p)=\{x\in M:\mathrm{d}(p,x)\le r\}$. Though the conclusion

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of Proposition 2 may be already known by some experts, it seems that it is still not well-known and there is no proof can be found in the earlier literature. That is the reason why I decided to write down a proof.

Remark 4. It can be proved that for $p \in M$ and $r > 0$, if the minimizing geodesic from p to each point x such that $\mathrm{d}(x, p) = r$ is unique, then the injectivity radius of $p \geq r$. However, the proof is more complicate than that of Proposition 2. So we will not go into that case here.

Proof of Proposition 2. Let $T_p^1 M$ denote the set of all unit vectors at p in M. Let us denote

$$A(p, r) = \{X \in T_p^1 M : \exp_p(tX) \text{ is minimal on } [0, r') \text{ for some } r' > r\}.$$

It suffices to show that $A(p, r)$ is open and close in $T_p^1 M$. Firstly, it is well-known that the function $\sigma : T_p^1 M \to \mathbb{R}^+$,

$$\sigma(X) = \sup\{t : \exp tX \text{ is minimal on } [0, t]\},$$

is continuous (see 2.1.5 Lemma in [1]). Hence by definition $A(p, r)$ is open.

Now let us show that $A(p, r)$ is closed in $T_p^1 M$. Assume a sequence of unit vectors $X_i \in A(p, r)$ converges to a unit vector $X \in T_p^1 M$, then the geodesic $\exp(tX_i)$ converges to $\exp(tX)$ point-wisely. Because all geodesic $\exp(tX_i)$ is minimal on $[0, r]$, the limit $\exp(tX)$ is also a minimal geodesic on $[0, r]$, and thus $\mathrm{d}(\exp(rX), p) = r$. Moreover, by the assumption of Proposition 2, $\exp(rX)$ is not a cut point of p. Hence, there is $\epsilon > 0$ such that $\exp(tX)$ is also minimal on $[0, r + \epsilon]$. Thus $X \in A(p, r)$ and $A(p, r)$ is closed.

Because $A(p, r)$ is open and closed, it coincides with $T_p^1 M$. Therefore the injectivity radius at p is $> r$. $\square$

The following corollaries directly follows from Proposition 2. Recall that p is called a pole if the injectivity radius of p is infinity. In particular, M is diffeomorphic to $\mathbb{R}^n$ by the exponential map $\exp_p : T_p M \to M$ at a pole.

Corollary 5. *Let M be a complete non-compact manifold. M possesses a pole at p if (and only if) there is a sequence $r_k \to \infty$ such that p is not a cut point of any point in $S(p, r_k)$.*

Corollary 6.

$$\limsup_{r \to \infty} \frac{i(p, r)}{r} > 1,$$

implies that every point in M is a pole. Hence either $\limsup_{r \to \infty} \frac{i(p,r)}{r} \in [0, 1]$, *or* $\limsup_{r \to \infty} \frac{i(p,r)}{r} = \infty$.

Because $\alpha(M) \leq \limsup_{r \to \infty} \frac{i(p,r)}{r}$, Corollary 6 not only answers Question 1, but also strength the homeomorphism result of Theorem 1.2 in [2] to diffeomorphism in the case of dimension 4.

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